Running title: *Modules and coalgebra Galois extensions*

ON MODULES ASSOCIATED TO COALGEBRA GALOIS EXTENSIONS

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Abstract

For a given entwining structure $(A,C)_\psi$ involving an algebra A, a coalgebra C, and an entwining map $\psi : C\otimes A\to A\otimes C$, a category $\mathbf{M}_A^C(\psi)$ of right $(A,C)_\psi$-modules is defined and its structure analysed. In particular, the notion of a measuring of $(A,C)_\psi$ to $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ is introduced, and certain functors between $\mathbf{M}_A^C(\psi)$ and $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ induced by such a measuring are defined. It is shown that these functors are inverse equivalences iff they are exact (or one of them faithfully exact) and the measuring satisfies a certain Galois-type condition. Next, left modules E and right modules $\bar{E}$ associated to a C-Galois extension A of B are defined. These can be thought of as objects dual to fibre bundles with coalgebra C in the place of a structure group, and a fibre V. Cross-sections of such associated modules are defined as module maps $E\to B$ or $\bar{E}\to B$. It is shown that they can be identified with suitably equivariant maps from the fibre to A. Also, it is shown that a C-Galois extension is cleft if and only if $A = B\otimes C$ as left B-modules and right C-comodules. The relationship between the modules E and $\bar{E}$ is studied in the case when V is finite-dimensional and in the case when the canonical entwining map is bijective.

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1. Introduction

The notion of a Hopf-Galois extension arose from the works of Chase and Sweedler [8] and Kreimer and Takeuchi [17] (see [23] for a review). From the geometric point of view, a Hopf-Galois extension is a dualisation of the notion of a principal bundle and thus it is a cornerstone of the Hopf algebra or quantum group gauge theory. Such a gauge theory, in the sense of connections, gauge transformations, curvature etc. on Hopf-Galois extensions was proposed in [3] and later developed in [15] [5] [1]. Also, the notion of a quantum fibre bundle as a module associated to the Hopf-Galois extension was introduced in [3]. This led to quantum group version of objects important in classical gauge theory such as sections of a vector bundle. Slightly different approaches to quantum group gauge theory, which take principal bundles as a framework of such a theory but do not use the Hopf-Galois extensions explicitly, were also proposed in [13], [24].

Motivated by the structure of quantum homogeneous spaces, the notion of a C-Galois extension A of an algebra B was recently introduced [4] as an object dual to a principal bundle on such a space. This has been done by requiring that A and C admit an *entwining structure* specified by a map $\psi : C \otimes A \to A \otimes C$ satisfying a set of (self-dual) conditions (cf. Definition 2.2). A gauge theory, in the above sense, on such a C-Galois extension was developed. In the present paper we derive the algebraic version of the classical correspondence between the gauge transformations (vertical automorphisms) and ad-covariant functions on a principal bundle (sections of an associated adjoint bundle). The main objective of the present paper, however, is to construct the algebraic counterpart of the notion of an associated fibre bundle - a "coalgebra fibre bundle". Our construction is motivated by a recent development of quantum and braided group Riemannian geometry in [20] and is a starting point for a more general coalgebra Riemannian geometry which is presented in [6]. The idea of the construction is to associate a certain B-module to a C-Galois extension of B and a C-comodule. There are two possibilities of associating such modules: they can be either left or right B-modules, depending on whether there is a left or right C-comodule involved (as opposed to the Hopf-Galois case, where the

similar construction leads to bimodules). We study both cases separately as well as the relationship between them. In both cases we derive the algebraic counterparts of the classical geometric equivalences between cross-sections and equivariant functions on a fibre bundle, and between cross-sections and trivialisations of a principal bundle, and thus we generalise the Hopf-Galois considerations of [1] to the C-Galois case. It turns out that to perform this analysis it is useful to consider the category $\mathbf{M}_A^C(\psi)$ of (right) $(A, C)_\psi$-modules. These are a natural generalisation of right (A, H)-Hopf modules. We introduce the notion of a measuring of entwining structures, and study when the functors between categories of entwined modules induced by such a measuring are inverse equivalences, thus extending the results of [7] proven for a generalisation of Hopf modules known as Doi-Hopf modules [10] [18].

Notation. We work over a ground field k. All algebras are associative and unital with the unit denoted by 1 (the unit map from k to the algebra is denoted by η). We use the standard algebra and coalgebra notation, i.e., Δ is a coproduct, μ is a product, ε is a counit, etc. The identity map from the space V to itself is also denoted by V. The unadorned tensor product stands for the tensor product over k. For an algebra A we denote by $\mathbf{M}_A$ (resp. ${}_A\mathbf{M}$) the category of right (resp. left) A-modules. For a right (resp. left) A-module V the action is denoted by μ_V (resp. ${}_V\mu$) whenether it needs to be specified as a map, or by a dot between elements. Similarly, for a coalgebra C we denote by $\mathbf{M}^C$ (resp. ${}^C\mathbf{M}$) the category of right (resp. left) C-comodules. A right (resp. left) coaction of C on V is denoted by Δ_V (resp. ${}_V\Delta$). Also, by ${}_A\mathbf{Mod}^C$ we denote the category of (A, C)-bimodules, *i.e.* left A-modules and right C-comodules V such that $\Delta_V \circ {}_V\mu = ({}_V\mu \otimes C) \circ (A \otimes \Delta_V)$, i.e., Δ_V is left A-linear. Similarly ${}^C\mathbf{Mod}_A$ is the category of right A-modules and left C-comodules with right A-linear coaction. The position of a subscript (resp. superscript) of Hom, End etc. indicates left or right module (resp. comodule) structure, e.g. Hom_{-A} are morphsisms in $\mathbf{M}_A$. For coactions and coproducts we use Sweedler's notation with suppressed summation sign: $\Delta_A(a) = a_{(0)} \otimes a_{(1)}$ for $a \in A \in \mathbf{M}^C$, ${}_V\Delta(v) = v_{(-1)} \otimes v_{(0)}$ for $v \in V \in {}^C\mathbf{M}$ and $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for $c \in C$. For

an algebra A and a coalgebra C we denote by $*$ the *convolution product* in $\mathrm{Hom}(C, A)$, i.e. $f * g(c) = f(c_{(1)})g(c_{(2)})$ for any $f, g \in \mathrm{Hom}(C, A)$ and $c \in C$. The convolution product makes $\mathrm{Hom}(C, A)$ into an associative algebra with unit $\eta \circ \varepsilon$. An element $f \in \mathrm{Hom}(C, A)$ is said to be *convolution invertible* if it is invertible with respect to $*$.

2. C-Galois extensions and their automorphisms

First recall the definition of a coalgebra Galois extension from [2]

Definition 2.1 *Let C be a coalgebra, A an algebra and a right C-comodule, and B a subalgebra of A, $B := \{b \in A \mid \forall a \in A\ \Delta_A(ba) = ba_{(0)} \otimes a_{(1)}\}$. We say that A is a* coalgebra Galois extension *(or C-Galois extension) of B iff the canonical left A-module right C-comodule map*

$$can := (\mu \otimes C) \circ (A \otimes_B \Delta_A) : A \otimes_B A \longrightarrow A \otimes C$$

is bijective. Such a C-Galois extension is denoted by $A(B)^C$.

Definition 2.1 generalises the notion of a Hopf-Galois extension (see [23] for a review). The latter is a C-Galois extension with $C = H$ being a Hopf algebra and A a right H-comodule algebra.

An important role in the analysis of coalgebra Galois extensions is played by the notion of an entwining structure [4] (closely connected with the theory of factorisation of algebras [19]).

Definition 2.2 *Let C be a coalgebra, A an algebra and let ψ be a k-linear map $\psi : C \otimes A \to A \otimes C$ such that*

$$\psi \circ (C \otimes \mu) = (\mu \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A), \qquad \psi \circ (C \otimes \eta) = \eta \otimes C, \tag{2.1}$$

$$(A \otimes \Delta) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A), \qquad (A \otimes \varepsilon) \circ \psi = \varepsilon \otimes A, \tag{2.2}$$

The triple (A, C, ψ) is called an entwining structure *and is denoted by $(A, C)_\psi$. The map ψ is called an* entwining map. *A morphism between entwining structures $(A, C)_\psi$ and $(\tilde{A}, \tilde{C})_{\tilde{\psi}}$ is a pair (f, g), where $f : A \to \tilde{A}$ is a unital algebra map and $g : C \to \tilde{C}$ is a countial coalgebra map such that $(f \otimes g) \circ \psi = \tilde{\psi} \circ (g \otimes f)$.*

Given an entwining structure $(A,C)_\psi$ we use the notation $\psi(c\otimes a) = a_\alpha\otimes c^\alpha$ (summation over a Greek index is understood), for all $a\in A$, $c\in C$.

For an entwining structure $(A,C)_\psi$, $\mathbf{M}_A^C(\psi)$ is the category of *right $(A,C)_\psi$-modules*. The objects of $\mathbf{M}_A^C(\psi)$ are right A-modules and right C-comodules M such that

$$\Delta_A(m\cdot a) = m_{(0)}\cdot\psi(m_{(1)}\otimes a) := m_{(0)}\cdot a_\alpha\otimes {m_{(1)}}^\alpha, \qquad \forall m\in M, a\in A \tag{2.3}$$

Morphisms in $\mathbf{M}_A^C(\psi)$ are right A-module right C-comodule maps.

It is shown in [2] that if $A(B)^C$ is a C-Galois extension, then $\psi : C\otimes A\to A\otimes C$, $\psi = can\circ(A\otimes_B\mu)\circ(\tau\otimes A)$ is a unique entwining map such that A is an object in $\mathbf{M}_A^C(\psi)$. Here $\tau : C\to A\otimes_B A$ is the *translation map*, i.e. $\tau(c) = can^{-1}(1\otimes c)$. This $(A,C)_\psi$ is called the *canonical entwining structure* associated to $A(B)^C$.

A C-Galois extension $A(B)^C$ is called a *cleft extension* iff there exists a convolution invertible, right C-comodule map $\Phi : C\to A$. Such a Φ is called a *cleaving map*. An object dual to a trivial principal bundle is an example of a cleft extension. The following proposition gives equivalent descriptions of cleft extensions, generalising [4, Proposition 2.9], where more structure on C and a different condition for Φ were assumed.

PROPOSITION 2.3 *Let C be a coalgebra, A be a right C-comodule and let B be as in Definition 2.1. If there exists a convolution invertible, right C-comodule map $\Phi : C\to A$ then the following are equivalent:*

(1) A is a C-Galois extension of B.

(2) There is an entwining structure $(A,C)_\psi$ such that $A\in\mathbf{M}_A^C(\psi)$ via Δ_A and μ.

(3) For every $a\in A$, $a_{(0)}\Phi^{-1}(a_{(1)})\in B$

If any of the above conditions hold, then $A\cong B\otimes C$ in ${}_B\mathbf{Mod}^C$.

Proof. (1) $\Rightarrow$ (2) follows from the result of [2], cited above.

(2) $\Rightarrow$ (3) Since $A\in\mathbf{M}_A^C(\psi)$, the coaction can be written as $\Delta_A(a) = 1_{(0)}\psi(1_{(1)}\otimes a)$, for all $a\in A$. The right C-colinearity of Φ together with the equality $1_{(0)}\varepsilon(c)\otimes 1_{(1)} = 1_{(0)}\psi(1_{(1)}\otimes\Phi(c_{(1)})\Phi^{-1}(c_{(2)}))$ and (2.1) imply that

$$\psi(c_{(1)}\otimes\Phi^{-1}(c_{(2)})) = \Phi^{-1}(c)\Delta_A(1). \tag{2.4}$$

Using this last equality and the fact that $A \in \mathbf{M}_A^C(\psi)$ one finds that (3) holds.

(3) $\Rightarrow$ (1) One easily verifies that the map $A \otimes C \to A \otimes_B A$, $a \otimes c \mapsto a\Phi^{-1}(c_{(1)}) \otimes_B \Phi(c_{(2)})$ is the inverse of *can*.

Finally, a ${}_B\mathbf{Mod}^C$ isomorphism $A \xrightarrow{\sim} B \otimes C$ is $a \mapsto a_{(0)}\Phi^{-1}(a_{(1)}) \otimes a_{(2)}$, and its inverse is $b \otimes c \mapsto b\Phi(c)$. $\square$

For $A(B)^C$, $\mathrm{Aut}(A(B)^C)$ denotes the group of left B-module, right C-comodule automorphisms of A, with the product $\mathcal{F}\mathcal{G} = \mathcal{G} \circ \mathcal{F}$, for all $\mathcal{F}, \mathcal{G} \in \mathrm{Aut}(A(B)^C)$. We now give a description of $\mathrm{Aut}(A(B)^C)$ which reflects the classical correspondence between gauge transformations of a principal bundle and ad-equivariant functions on it (cf. [16, 7.1.6])[2].

THEOREM 2.4 *The group* $\mathrm{Aut}(A(B)^C)$ *is isomorphic to the group* $\mathfrak{C}(A)$ *of convolution invertible maps* $f : C \to A$ *such that*

$$\psi \circ (C \otimes f) \circ \Delta = (f \otimes C) \circ \Delta, \tag{2.5}$$

where ψ *is the canonical entwining map. The product in* $\mathfrak{C}(A)$ *is the convolution product.*

Proof. We use the π-method of [12]. Applying the functor $\mathrm{Hom}_{A-}(-, A)$ to $can : A \otimes_B A \xrightarrow{\sim} A \otimes C$ one obtains the isomorphism $\pi : \mathrm{Hom}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{B-}(A, A)$. Explicitly, $\pi(f) = \mu \circ (A \otimes f) \circ \Delta_A$ and $\pi^{-1}(\mathcal{F}) = \mu \circ (A \otimes_B \mathcal{F}) \circ \tau$, for all $f \in \mathrm{Hom}(C, A)$, $\mathcal{F} \in \mathrm{Hom}_{B-}(A, A)$. Since $A \in \mathbf{M}_A^C(\psi)$ we have $\Delta_A(\pi(f)(a)) = \Delta_A(a_{(0)}f(a_{(1)})) = a_{(0)}\psi(a_{(1)} \otimes f(a_{(2)}))$. It means that $\pi(f)$ is right C-colinear if and only if for all $a \in A$, $a_{(0)}\psi(a_{(1)} \otimes f(a_{(2)})) = a_{(0)}f(a_{(1)}) \otimes a_{(2)}$. Using the definition of τ one easily sees that this is equivalent to (2.5). Next take any $f, g \in \mathfrak{C}(A)$. Since $\pi(f)$ is a right C-comodule map, we find for all $a \in A$

$$(\pi(g) \circ \pi(f))(a) = \pi(f)(a_{(0)})g(a_{(1)}) = a_{(0)}f(a_{(1)})g(a_{(2)}) = \pi(f * g)(a).$$

[2] In the Hopf-Galois case, (2.5) means that $f \in \mathrm{Hom}^{-C}(C, A)$, where C is in $\mathbf{M}^C$ via the right adjoint coaction. Also, a condition similar to (2.5) characterises connection one-forms on $A(B)^C$.

Finally, if f satisfies (2.5) then, using (2.1), for all $c \in C$ one has

$$c_{(1)} \otimes 1 \otimes c_{(2)} = c_{(1)} \otimes \psi(c_{(2)} \otimes f(c_{(3)}) f^{-1}(c_{(4)})) = c_{(1)} \otimes f(c_{(2)}) \psi(c_{(3)} \otimes f^{-1}(c_{(4)})).$$

Applying $f^{-1} \otimes A \otimes C$ to the above equality and multiplying the first two factors one finds that f^{-1} satisfies (2.5). Also $\eta \circ \varepsilon$ satisfies (2.5). Therefore $\mathfrak{C}(A)$ is a group with respect to the convolution product as claimed, and $\mathrm{Aut}(A(B)^C) \cong \mathfrak{C}(A)$ as groups. $\square$

Finally we notice that if $A(B)^C$ is a cleft C-Galois extension, then since $A \cong B \otimes C$ in ${}_B\mathbf{Mod}^C$, there is an algebra isomorphism $\mathrm{End}_{B-}^{-C}(A) \cong \mathrm{Hom}(C, B)^{op}$ (cf. [23, Lemma on p.91]). This implies that, in this case, the group $\mathrm{Aut}(A(B)^C)$ and therefore also $\mathfrak{C}(A)$ are isomorphic to the group of convolution invertible maps $C \to B$. This reflects the classical description of local gauge transformations (cf. [16, 7.1.7]).

3. The structure of $(A, C)_\psi$-modules

In this section we analyse the structure of the category $\mathbf{M}_A^C(\psi)$ of $(A, C)_\psi$-modules, i.e. right A-modules and right C-comodules characterised by (2.3). This category can be viewed as a generalisation of the categories well-studied in the Hopf algebra theory.

Example 3.1 (1) Let $C = H$ be a Hopf algebra, A be a right H-comodule algebra and let $\psi : H \otimes A \to A \otimes H$ be defined by $\psi : h \otimes a \mapsto a_{(0)} \otimes h a_{(1)}$. Then $\mathbf{M}_A^H(\psi)$ is the category of right (A, H)-Hopf modules introduced in [9].

(2) Let $A = C = H$ be a Hopf algebra and let the entwining map $\psi : H \otimes H \to H \otimes H$ be given by $\psi : g \otimes h \mapsto h_{(2)} \otimes (S h_{(1)}) g h_{(3)}$, where S is the antipode in H. Then $\mathbf{M}_H^H(\psi)$ is the category of right-right Yetter-Drinfeld modules introduced in [30], [25].

(3) Examples (1) and (2) are special cases of the following construction. Let H be a Hopf algebra, A be a right H-comodule algebra and C a right H-module coalgebra. Then $(A, C)_\psi$ is an entwining structure with $\psi : c \otimes a \mapsto a_{(0)} \otimes c \cdot a_{(1)}$ and $\mathbf{M}_A^C(\psi)$ is the category of unifying Hopf modules (or Doi-Hopf modules) introduced in [10] [18]. A special case of this category with $C = H/I$ a quotient coalgebra and a quotient right H-module was considered in [26] (in particular, $\mathbf{M}_H^C(\psi)$ for the canonical entwining structure $(H, C)_\psi$

associated to a C-Galois extension $H(B)^C$, where B is a quantum homogeneous space of H (cf. [4, Example 2.5] or [28, Lemma 1.3.]), is of this type).

Definition 3.2 *Let $(A,C)_\psi$ and $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ be entwining structures, and let $\alpha:\tilde{C}\otimes\tilde{A}\to A$, $\gamma:\tilde{C}\to A\otimes C$ be linear maps such that*

$$\mu\circ(\alpha\otimes\alpha)\circ(\tilde{C}\otimes\tilde{\psi}\otimes\tilde{A})\circ(\tilde{\Delta}\otimes\tilde{A}\otimes\tilde{A})=\alpha\circ(\tilde{C}\otimes\tilde{\mu}),\qquad \alpha\circ(\tilde{C}\otimes\tilde{\eta})=\eta\circ\tilde{\varepsilon},\quad(3.6)$$

$$(\mu\otimes C\otimes C)\circ(A\otimes\psi\otimes C)\circ(\gamma\otimes\gamma)\circ\tilde{\Delta}=(A\otimes\Delta)\circ\gamma,\qquad (A\otimes\varepsilon)\circ\gamma=\eta\circ\tilde{\varepsilon},\quad(3.7)$$

$$(\mu\otimes C)\circ(\alpha\otimes\gamma)\circ(\tilde{C}\otimes\tilde{\psi})\circ(\tilde{\Delta}\otimes\tilde{A})=(\mu\otimes C)\circ(A\otimes\psi)\circ(\gamma\otimes\alpha)\circ(\tilde{\Delta}\otimes\tilde{A}),\quad(3.8)$$

where non-tilded (tilded) structure maps correspond to A,C $(\tilde{A},\tilde{C})$. The pair (α,γ) is said to measure *$(\tilde{A},\tilde{C})_{\tilde{\psi}}$ to $(A,C)_\psi$. Such a measuring is denoted by $(\tilde{A},\tilde{C})_{\tilde{\psi}}\models\frac{\alpha}{\gamma}(A,C)_\psi$.*

The terminology of Definition 3.2 is motivated by the fact that if one chooses ψ and $\tilde{\psi}$ to be the twists, and $C=k$, then the pair $(\alpha,\eta\circ\tilde{\varepsilon})$ measures $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ to $(A,k)_\psi$ iff $(\alpha,\tilde{C})$ measures $\tilde{A}$ to A in the sense of [29, p. 138]. If (f,g) is a morphism from $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ to $(A,C)_\psi$ then $(\tilde{\varepsilon}\otimes f,\eta\otimes g)$ measures $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ to $(A,C)_\psi$.

Proposition 3.3 *Let (α,γ) measure $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ to $(A,C)_\psi$. Then:*

(1) For all $M\in\mathbf{M}_A$, $M\otimes\tilde{C}$ is an $(\tilde{A},\tilde{C})_{\tilde{\psi}}$-module via $\Delta_{M\otimes\tilde{C}}=M\otimes\tilde{\Delta}$ and $\mu_{M\otimes\tilde{C}}=(\mu_M\otimes\tilde{C})\circ(M\otimes\alpha\otimes\tilde{C})\circ(M\otimes\tilde{C}\otimes\tilde{\psi})\circ(M\otimes\tilde{\Delta}\otimes\tilde{A})$.

(2) For all $\tilde{M}\in\mathbf{M}^{\tilde{C}}$, $\tilde{M}\otimes A$ is an $(A,C)_\psi$-module via $\mu_{\tilde{M}\otimes A}=\tilde{M}\otimes\mu$ and $\Delta_{\tilde{M}\otimes A}=(\tilde{M}\otimes\mu\otimes C)\circ(\tilde{M}\otimes A\otimes\psi)\circ(\tilde{M}\otimes\gamma\otimes A)\circ(\Delta_{\tilde{M}}\otimes A)$.

Proof. (1) We first show that $\mu_{M\otimes\tilde{C}}$ (later denoted by a dot) is an action of $\tilde{A}$ on $M\otimes\tilde{C}$. Explicitly, for any $m\in M$, $\tilde{c}\in\tilde{C}$ and $\tilde{a}\in\tilde{A}$ this map is $(m\otimes\tilde{c})\cdot\tilde{a}=m\cdot\alpha(\tilde{c}_{(1)}\otimes\tilde{a}_\beta)\otimes{\tilde{c}_{(2)}}^\beta$. By the second of equations (3.6) and (2.1) we have that $(m\otimes\tilde{c})\cdot 1=m\otimes\tilde{c}$. Furthermore, for any $\tilde{a}'\in\tilde{A}$,

$$((m\otimes\tilde{c})\cdot\tilde{a})\cdot\tilde{a}' \;=\; m\cdot\alpha(\tilde{c}_{(1)}\otimes\tilde{a}_\beta)\alpha({{\tilde{c}_{(2)}}^\beta}_{(1)}\otimes\tilde{a}'_\delta)\otimes{{{\tilde{c}_{(2)}}^\beta}_{(2)}}^\delta$$

$$
\begin{aligned}
&= m\cdot\alpha(\tilde{c}_{(1)}\otimes\tilde{a}_{\beta\lambda})\alpha(\tilde{c}_{(2)}{}^{\lambda}\otimes\tilde{a}'_{\delta})\otimes\tilde{c}_{(3)}{}^{\beta\delta} && \text{(by (2.2))}\\
&= m\cdot\alpha(\tilde{c}_{(1)}\otimes\tilde{a}_{\beta}\tilde{a}'_{\delta})\otimes\tilde{c}_{(2)}{}^{\beta\delta} && \text{(by (3.6))}\\
&= (m\otimes\tilde{c})\cdot(\tilde{a}\tilde{a}') && \text{(by (2.1))}.
\end{aligned}
$$

Clearly, $\Delta_{M\otimes\tilde{C}}$ is a right coaction of $\tilde{C}$ on $M\otimes\tilde{C}$. For any $m\in M$, $\tilde{c}\in\tilde{C}$, $\tilde{a}\in\tilde{A}$,

$$
\begin{aligned}
\Delta_{M\otimes\tilde{C}}((m\otimes\tilde{c})\cdot\tilde{a}) &= m\cdot\alpha(\tilde{c}_{(1)}\otimes\tilde{a}_{\beta})\otimes\tilde{c}_{(2)}{}^{\beta}{}_{(1)}\otimes\tilde{c}_{(2)}{}^{\beta}{}_{(2)}\\
&= m\cdot\alpha(\tilde{c}_{(1)}\otimes\tilde{a}_{\beta\delta})\otimes\tilde{c}_{(2)}{}^{\delta}\otimes\tilde{c}_{(3)}{}^{\beta} \qquad \text{(by (2.2))}\\
&= (m\otimes\tilde{c}_{(1)})\cdot\tilde{a}_{\beta}\otimes\tilde{c}_{(2)}{}^{\beta}.
\end{aligned}
$$

This proves that $M\otimes\tilde{C}$ is an object in $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$.

(2) Dual to (1). □

Corollary 3.4 *Let $(A,C)_{\psi}$ be an entwining structure. Then:*

(1) For any right A-module M, $M\otimes C$ is an $(A,C)_{\psi}$-module with the action $m\otimes c\otimes a\mapsto m\cdot\psi(c\otimes a)$ and the coaction $\Delta_{M\otimes C}=M\otimes\Delta$.

(2) For any right C-comodule V, $V\otimes A$ is an $(A,C)_{\psi}$-module with the action $V\otimes\mu$ and the coaction $v\otimes a\mapsto v_{(0)}\otimes\psi(v_{(1)}\otimes a)$.

Proof. To prove (1) take $(A,k)_{\sigma}$, where $\sigma: k\otimes A\to A\otimes k$ is a twist (canonically equivalent to the map A) and notice that $(\varepsilon\otimes A,\eta\circ\varepsilon)$, measures $(A,C)_{\psi}$ to $(A,k)_{\sigma}$. Then Proposition 3.3(1) yields the assertion. Statement (2) is dual to (1), and can be deduced from Proposition 3.3(2) by taking $(\tilde{A},\tilde{C})_{\tilde{\psi}}=(k,C)_{\sigma}$ and $(\alpha,\gamma)=(\eta\circ\varepsilon,\eta\otimes C)$. □

Proposition 3.5 *Let (α,γ) measure $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ to $(A,C)_{\psi}$. Then:*

(1) For all $M\in\mathbf{M}_A^C(\psi)$ the map $\hat{\ell}^M: M\otimes\tilde{C}\to M\otimes C\otimes\tilde{C}$,

$$
\hat{\ell}^M := \Delta_M\otimes\tilde{C}-(\mu_M\otimes C\otimes\tilde{C})\circ(M\otimes\gamma\otimes\tilde{C})\circ(M\otimes\tilde{\Delta})
$$

is a morphism in $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$, where $M\otimes\tilde{C}$ and $M\otimes C\otimes\tilde{C}$ are viewed in $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ as in Proposition 3.3(1) with $M\otimes C\in\mathbf{M}_A$ as in Corollary 3.4(1).

(2) For all $\tilde{M} \in \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$, the map $\hat{\ell}_{\tilde{M}} : \tilde{M} \otimes \tilde{A} \otimes A \to \tilde{M} \otimes A$,

$$\hat{\ell}_{\tilde{M}} = \mu_{\tilde{M}} \otimes A - (\tilde{M} \otimes \mu) \circ (\tilde{M} \otimes \alpha \otimes A) \circ (\Delta_{\tilde{M}} \otimes \tilde{A} \otimes A)$$

is a morphism in $\mathbf{M}_A^C(\psi)$, where $\tilde{M} \otimes A$ and $\tilde{M} \otimes \tilde{A} \otimes A$ are viewed in $\mathbf{M}_A^C(\psi)$ as in Proposition 3.3(2) with $\tilde{M} \otimes \tilde{A} \in \mathbf{M}^{\tilde{C}}$ as in Corollary 3.4(2).

Proof. We introduce the Sweedler-like notation $\gamma(\tilde{c}) = \tilde{c}^{[1]} \otimes \tilde{c}^{[2]}$ (summation understood). With this notation the map $\hat{\ell}^M$ explicitly reads for all $m \in M$, $\tilde{c} \in \tilde{C}$, $\hat{\ell}^M(m \otimes \tilde{c}) = m_{(0)} \otimes m_{(1)} \otimes \tilde{c} - m \cdot \tilde{c}_{(1)}{}^{[1]} \otimes \tilde{c}_{(1)}{}^{[2]} \otimes \tilde{c}_{(2)}$. Clearly, $\hat{\ell}^M$ is a right $\tilde{C}$-comodule map. Next we have

$$\begin{aligned}
\hat{\ell}^M((m \otimes \tilde{c}) \cdot \tilde{a}) &= \hat{\ell}^M(m \cdot \alpha(\tilde{c}_{(1)} \otimes \tilde{a}_\beta) \otimes \tilde{c}_{(2)}{}^\beta) \\
&= m_{(0)} \cdot \alpha(\tilde{c}_{(1)} \otimes \tilde{a}_\beta)_\delta \otimes m_{(1)}{}^\delta \otimes \tilde{c}_{(2)}{}^\beta \qquad (M \in \mathbf{M}_A^C(\psi)) \\
&\quad - m \cdot \alpha(\tilde{c}_{(1)} \otimes \tilde{a}_\beta) \tilde{c}_{(2)}{}^\beta{}_{(1)}{}^{[1]} \otimes \tilde{c}_{(2)}{}^\beta{}_{(1)}{}^{[2]} \otimes \tilde{c}_{(2)}{}^\beta{}_{(2)} \\
&= m_{(0)} \cdot \alpha(\tilde{c}_{(1)} \otimes \tilde{a}_\beta)_\delta \otimes m_{(1)}{}^\delta \otimes \tilde{c}_{(2)}{}^\beta \\
&\quad - m \cdot \alpha(\tilde{c}_{(1)} \otimes \tilde{a}_{\beta\delta}) \tilde{c}_{(2)}{}^{\delta[1]} \otimes \tilde{c}_{(2)}{}^{\delta[2]} \otimes \tilde{c}_{(3)}{}^\beta \qquad (\text{by } (2.2)) \\
&= m_{(0)} \cdot \alpha(\tilde{c}_{(1)} \otimes \tilde{a}_\beta)_\delta \otimes m_{(1)}{}^\delta \otimes \tilde{c}_{(2)}{}^\beta \\
&\quad - m \cdot \tilde{c}_{(1)}{}^{[1]} \alpha(\tilde{c}_{(2)} \otimes \tilde{a}_\beta)_\delta \otimes \tilde{c}_{(1)}{}^{[2]\delta} \otimes \tilde{c}_{(3)}{}^\beta \qquad (\text{by } (3.8)) \\
&= (m_{(0)} \otimes m_{(1)}) \cdot \alpha(\tilde{c}_{(1)} \otimes \tilde{a}_\beta) \otimes \tilde{c}_{(2)}{}^\beta \\
&\quad - (m \cdot \tilde{c}_{(1)}{}^{[1]} \otimes \tilde{c}_{(1)}{}^{[2]}) \cdot \alpha(\tilde{c}_{(2)} \otimes \tilde{a}_\beta) \otimes \tilde{c}_{(3)}{}^\beta \\
&= (m_{(0)} \otimes m_{(1)} \otimes \tilde{c}) \cdot \tilde{a} - (m \cdot \tilde{c}_{(1)}{}^{[1]} \otimes \tilde{c}_{(1)}{}^{[2]} \otimes \tilde{c}_{(2)}) \cdot \tilde{a} \\
&= \hat{\ell}^M(m \otimes \tilde{c}) \cdot \tilde{a}.
\end{aligned}$$

To derive the fifth and the sixth equations we used definitions of actions of A on $M \otimes C$ in Corollary 3.4(1) and of $\tilde{A}$ on $M \otimes C \otimes \tilde{C}$ in Proposition 3.3(1) combined with Corollary 3.4(1). This completes the proof that $\hat{\ell}^M$ is a morphism in $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$.

(2) Dual to (1). □

Given a measuring $(\tilde{A},\tilde{C})_{\tilde{\psi}}|\frac{\alpha}{\gamma}(A,C)_\psi$, for all $M \in \mathbf{M}_A^C(\psi)$ define $M\widehat{\Box}_C\tilde{C} \subseteq M \otimes \tilde{C}$ via the exact sequence

$$0 \longrightarrow M\widehat{\Box}_C\tilde{C} \longrightarrow M \otimes \tilde{C} \xrightarrow{\hat{\ell}^M} M \otimes C \otimes \tilde{C}.$$

Since the above sequence is a sequence in $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$, $M\widehat{\Box}_C\tilde{C}$ is an $(\tilde{A},\tilde{C})_{\tilde{\psi}}$-module via the restriction of the structure maps in Proposition 3.3(1). Thus we obtain a functor $-\widehat{\Box}_C\tilde{C} : \mathbf{M}_A^C(\psi) \to \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$. Dually, for all $\tilde{M} \in \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ define $\tilde{M}\hat{\otimes}_{\tilde{A}}A$ by the exact sequence

$$\tilde{M} \otimes \tilde{A} \otimes A \xrightarrow{\hat{\ell}_{\tilde{M}}} \tilde{M} \otimes A \xrightarrow{\hat{\pi}_{\tilde{M}}} \tilde{M}\hat{\otimes}_{\tilde{A}}A \longrightarrow 0$$

(if there is no need to specify the module $\tilde{M}$ we will write $\hat{\pi}$ for $\hat{\pi}_{\tilde{M}}$). $\tilde{M}\hat{\otimes}_{\tilde{A}}A$ is an $(A,C)_\psi$-module with the structure maps obtained from the structure maps in Proposition 3.3(2), by projecting through $\hat{\pi}_M$. Thus we have the functor $-\hat{\otimes}_{\tilde{A}}A : \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi}) \to M_A^C(\psi)$.

Proposition 3.6 *Given a measuring $(\tilde{A},\tilde{C})_{\tilde{\psi}}|\frac{\alpha}{\gamma}(A,C)_\psi$, the functor $-\widehat{\Box}_C\tilde{C} : \mathbf{M}_A^C(\psi) \to \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ is the right adjoint of the functor $-\hat{\otimes}_{\tilde{A}}A$.*

Proof. We claim that for all $M \in \mathbf{M}_A^C(\psi)$, $\tilde{M} \in \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ there is a natural isomorphism $\zeta_{\tilde{M},M} : \mathrm{Hom}_{-A}^{-C}(\tilde{M}\hat{\otimes}_{\tilde{A}}A, M) \to \mathrm{Hom}_{-\tilde{A}}^{-\tilde{C}}(\tilde{M}, M\widehat{\Box}_C\tilde{C})$, $\zeta_{\tilde{M},M} : f \mapsto (f \circ \hat{\pi}\otimes\tilde{C}) \circ (\tilde{M} \otimes \eta \otimes \tilde{C}) \circ \Delta_{\tilde{M}}$. Explicitly for all $\tilde{m} \in \tilde{M}$, $\zeta_{\tilde{M},M}(f)(\tilde{m}) = f(\hat{\pi}(\tilde{m}_{(0)} \otimes 1)) \otimes \tilde{m}_{(1)}$. The output of $\zeta_{\tilde{M},M}(f)$ is in $M\widehat{\Box}_C\tilde{C}$, since

$$\begin{aligned}
\hat{\ell}^M(\zeta_{\tilde{M},M}(f)(\tilde{m})) &= \Delta_M(f(\hat{\pi}(\tilde{m}_{(0)} \otimes 1))) \otimes \tilde{m}_{(1)} - f(\hat{\pi}(\tilde{m}_{(0)} \otimes 1)) \cdot \tilde{m}_{(1)}{}^{[1]} \otimes \tilde{m}_{(1)}{}^{[2]} \otimes \tilde{m}_{(2)} \\
&= f(\hat{\pi}(\tilde{m}_{(0)} \otimes 1)_{(0)}) \otimes \hat{\pi}(\tilde{m}_{(0)} \otimes 1)_{(1)} \otimes \tilde{m}_{(1)} \\
&\quad - f(\hat{\pi}(\tilde{m}_{(0)} \otimes 1) \cdot \tilde{m}_{(1)}{}^{[1]}) \otimes \tilde{m}_{(1)}{}^{[2]} \otimes \tilde{m}_{(2)} \\
&= f(\hat{\pi}(\tilde{m}_{(0)} \otimes \tilde{m}_{(1)}{}^{[1]})) \otimes \tilde{m}_{(1)}{}^{[2]} \otimes \tilde{m}_{(2)} \\
&\quad - f(\hat{\pi}(\tilde{m}_{(0)} \otimes \tilde{m}_{(1)}{}^{[1]})) \otimes \tilde{m}_{(1)}{}^{[2]} \otimes \tilde{m}_{(2)} \\
&= 0,
\end{aligned}$$

where we used the definition of $\hat{\ell}^M$ to derive the first equality, then the fact that f is a morphism in $\mathbf{M}_A^C(\psi)$ to obtain the second one. The third equality was obtained by using

the explicit form of the coaction of C on $\tilde{M}\hat{\otimes}_{\tilde{A}}A$ and the fact that $\hat{\pi}$ is a right A-module map.

It is clear that $\zeta_{\tilde{M},M}(f)$ is a right $\tilde{C}$-comodule map, it is also right $\tilde{A}$-linear since

$$
\begin{aligned}
\zeta_{\tilde{M},M}(f)(\tilde{m})\cdot\tilde{a} &= f(\hat{\pi}(\tilde{m}_{(0)}\otimes 1))\cdot\alpha(\tilde{m}_{(1)}\otimes\tilde{a}_{\beta})\otimes{\tilde{m}_{(2)}}^{\beta} \\
&= f(\hat{\pi}(\tilde{m}_{(0)}\otimes\alpha(\tilde{m}_{(1)}\otimes\tilde{a}_{\beta})))\otimes{\tilde{m}_{(2)}}^{\beta} && (f \text{ is right } A\text{-linear}) \\
&= f(\hat{\pi}(\tilde{m}_{(0)}\cdot\tilde{a}_{\beta}\otimes 1))\otimes{\tilde{m}_{(1)}}^{\beta} && (\text{by definition of } \tilde{M}\hat{\otimes}_{\tilde{A}}A) \\
&= \zeta_{\tilde{M},M}(f)(\tilde{m}\cdot\tilde{a}) && (\tilde{M}\in\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi}))
\end{aligned}
$$

It is an easy exercise to verify that $\zeta_{\tilde{M},M}$ is natural in $\tilde{M}$ and M and that its inverse is $\zeta_{\tilde{M},M}^{-1}(g)\circ\hat{\pi} = \mu_M\circ(M\otimes\tilde{\varepsilon}\otimes A)\circ(g\otimes A)$. $\square$

Corollary 3.7 *Let $(A,C)_{\psi}$ be an entwining structure. Then:*

(1) The functor $-\otimes C:\mathbf{M}_A\to\mathbf{M}_A^C(\psi)$ is the right adjoint of the forgetful functor $\mathbf{M}_A^C(\psi)\to\mathbf{M}_A$.

(2) The functor $-\otimes A:\mathbf{M}^C\to\mathbf{M}_A^C(\psi)$ is the left adjoint of the forgetful functor $\mathbf{M}_A^C(\psi)\to\mathbf{M}^C$.

Proof. To prove (1) take the measuring in the proof of Corollary 3.4(1). Then $\mathbf{M}_A^k(\sigma) = \mathbf{M}_A$, and for all $M\in\mathbf{M}_A$, $M\widehat{\square}_k C = M\otimes C$, while for all $N\in\mathbf{M}_A^C(\psi)$, $N\hat{\otimes}_A A = N$. Now Proposition 3.6 yields the assertion. Statement (2) is dual to (1), and can be deduced from Proposition 3.6 by taking $(\tilde{A},\tilde{C})_{\tilde{\psi}} = (k,C)_{\sigma}$ and $(\alpha,\gamma) = (\eta\circ\varepsilon,\eta\otimes C)$. $\square$

From the proof of Proposition 3.6 it is clear that the adjunction morphisms are $\Psi_{\tilde{M}} = (\hat{\pi}_{\tilde{M}}\otimes\tilde{C})\circ(\tilde{M}\otimes\eta\otimes\tilde{C})\circ\Delta_{\tilde{M}}:\tilde{M}\to(\tilde{M}\hat{\otimes}_{\tilde{A}}A)\widehat{\square}_C\tilde{C}$, and $\Phi_M:(M\widehat{\square}_C\tilde{C})\hat{\otimes}_{\tilde{A}}A\to M$ determined by $\Phi_M\circ\hat{\pi}_{M\widehat{\square}_C\tilde{C}} = \mu_M\circ(M\otimes\tilde{\varepsilon}\otimes A)$, for all $M\in\mathbf{M}_A^C(\psi)$ and $\tilde{M}\in\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$.

Definition 3.8 (cf. Definition 1.4 in [7]) $(\tilde{A},\tilde{C})_{\tilde{\psi}}\models^{\alpha}_{\gamma}(A,C)_{\psi}$ *is said to be a* Galois measuring *if the adjunctions $\Psi_{\tilde{C}\otimes\tilde{A}}$, $\Phi_{A\otimes C}$ are bijective ($C\otimes A\in\mathbf{M}_A^C(\psi)$ and $\tilde{A}\otimes\tilde{C}\in\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ as in Corollary 3.4).*

Example 3.9 Assume that A is an $(A,C)_\psi$-module and let $B := \{b \in A \mid \forall a \in A,\ \Delta_A(ba) = b\Delta_A(a)\}$. Take the trivial entwining structure $(B,k)_\sigma$, so that $\mathbf{M}_B^k(\sigma) = \mathbf{M}_B$. Then $(\iota_B, \Delta_A \circ \eta)$, where $\iota_B : B \hookrightarrow A$ is the canonical inclusion, measures $(B,k)_\sigma$ to $(A,C)_\psi$. With this measuring, for all $V \in \mathbf{M}_B$, $V\hat{\otimes}_B A = V \otimes_B A$, while for all $M \in \mathbf{M}_A^C(\psi)$, $M\widehat{\Box}_C k = M_0 := \{m \in M \mid \Delta_M(m) = m \cdot 1_{(0)} \otimes 1_{(1)}\}$. Notice that $M_0 = \{m \in M \mid \forall a \in A,\ \Delta_M(m \cdot a) = m \cdot a_{(0)} \otimes a_{(1)}\}$. Indeed, if $m \in M_0$ then

$$\Delta_M(m \cdot a) = m_{(0)} \cdot \psi(m_{(1)} \otimes a) = m \cdot (1_{(0)}\psi(1_{(1)} \otimes a)) = m \cdot a_{(0)} \otimes a_{(1)},$$

since $\Delta_A(a) = 1_{(0)}\psi(1_{(1)} \otimes a)$. In particular $B = A_0$, so that $\Psi_{B\otimes k} = B$. On the other hand $A \cong (A\otimes C)_0$ via $a \mapsto a1_{(0)} \otimes 1_{(1)}$, $\sum_i a^i \otimes c^i \mapsto \sum_i \varepsilon(c^i)a^i$. Taking this isomorphism into account we have $\Phi_{A\otimes C} = can$, and we conclude that $(B,k)_\sigma|\frac{\iota_B}{\Delta_A\circ\eta}(A,C)_\psi$ is Galois iff the extension $B \hookrightarrow A$ is Galois (with the canonical entwining map ψ).

Theorem 3.10 *Let (α,γ) measure $(\tilde{A},\tilde{C})_{\tilde{\psi}}$ to $(A,C)_\psi$. Then the following are equivalent:*

(1) The functors $-\widehat{\Box}_C\tilde{C}$, $-\hat{\otimes}_{\tilde{A}}A$ are inverse equivalences.

(2) The functors $-\widehat{\Box}_C\tilde{C}$, $-\hat{\otimes}_{\tilde{A}}A$ are exact and $(\tilde{A},\tilde{C})_{\tilde{\psi}}|\frac{\alpha}{\gamma}(A,C)_\psi$ is Galois.

(3) The functor $-\hat{\otimes}_{\tilde{A}}A$ is faithfully exact and $(\tilde{A},\tilde{C})_{\tilde{\psi}}|\frac{\alpha}{\gamma}(A,C)_\psi$ is Galois.

(4) The functor $-\widehat{\Box}_C\tilde{C}$ is faithfully exact and $(\tilde{A},\tilde{C})_{\tilde{\psi}}|\frac{\alpha}{\gamma}(A,C)_\psi$ is Galois.

Proof. Recall that a functor is exact (resp. faithfully exact) if it preserves (resp. preserves and reflects) exact sequences. (1) clearly implies (2), (3) and (4). To show that (2) implies (1), first notice that $(A\otimes C)\widehat{\Box}_C\tilde{C} \cong A \otimes \tilde{C}$ in ${}_A\mathbf{Mod}^{\tilde{C}}$ via $A \otimes \varepsilon \otimes \tilde{C}$ and $(\mu \otimes C \otimes \tilde{C}) \circ (A \otimes \gamma \otimes C) \circ (A \otimes \tilde{\Delta})$. Let $\tilde{M} = (A\otimes C)\widehat{\Box}_C\tilde{C}$. Then for all $M \in \mathbf{M}_A^C(\psi)$, $M \otimes_A \tilde{M}$ is in $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ via $M \otimes_A \mu_{\tilde{M}}$, $M \otimes_A \Delta_{\tilde{M}}$ and there is a commutative diagram:

$$\begin{tikzcd}
(M\otimes_A\tilde{M})\otimes\tilde{A}\otimes A \arrow[r] \arrow[d, "\cong"] & (M\otimes_A\tilde{M})\otimes A \arrow[r, "\hat{\pi}_{M\otimes\tilde{M}}"] \arrow[d, "\cong"] & (M\otimes_A\tilde{M})\hat{\otimes}_{\tilde{A}}A \arrow[r] \arrow[d, "f"] & 0 \\
M\otimes_A(\tilde{M}\otimes\tilde{A}\otimes A) \arrow[r] & M\otimes_A(\tilde{M}\otimes A) \arrow[r, "M\otimes_A\hat{\pi}_{\tilde{M}}"] & M\otimes_A(\tilde{M}\hat{\otimes}_{\tilde{A}}A) \arrow[r] & 0
\end{tikzcd}$$

The top row is exact since it is the defining sequence of $\hat{\otimes}$. The bottom row is the defining sequence of $\hat{\otimes}$ tensored with M and thus is exact since the tensor product is right exact. Therefore the map f (constructed from the diagram) is an isomorphism, and we have:

$$(M \otimes \tilde{C})\hat{\otimes}_{\tilde{A}} A \cong (M \otimes_A ((A \otimes C)\widehat{\Box}_C \tilde{C}))\hat{\otimes}_{\tilde{A}} A \cong M \otimes_A (((A \otimes C)\widehat{\Box}_C \tilde{C})\hat{\otimes}_{\tilde{A}} A).$$

Thus we can consider the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & (M\widehat{\Box}_C\tilde{C})\hat{\otimes}_{\tilde{A}}A & \longrightarrow & (M\otimes\tilde{C})\hat{\otimes}_{\tilde{A}}A & \longrightarrow & (M\otimes C\otimes\tilde{C})\hat{\otimes}_{\tilde{A}}A & \\ & \downarrow{\scriptstyle \Phi_M} & & \downarrow{\scriptstyle M\otimes_A\Phi_{A\otimes C}} & & \downarrow{\scriptstyle (M\otimes C)\otimes_A\Phi_{A\otimes C}} & (3.9) \\ 0 \longrightarrow & M & \xrightarrow{\Delta_M} & M\otimes C & \xrightarrow{\ell_{MC}} & M\otimes C\otimes C & \end{array}$$

The top row is the defining sequence of $M\widehat{\Box}_C\tilde{C}$ acted upon by $-\hat{\otimes}_{\tilde{A}}A$ and thus is exact by the exactness of $-\hat{\otimes}_{\tilde{A}}A$. In the bottom row, $\ell_{MC} = \Delta_M \otimes C - M \otimes \Delta$ and hence the sequence is exact by the definition of the coproduct. Since (α, γ) is a Galois measuring, the maps in the second and the third columns are bijective and thus so is Φ_M.

Now, reversing the arrows in the above diagram, interchanging $\widehat{\Box}$ with $\hat{\otimes}$, A with C, tilded expressions with the non-tilded ones, coactions with actions, and Φ with Ψ one obtains the diagram from which one deduces that also $\Psi_{\tilde{M}}$ is bijective provided the functor $-\widehat{\Box}_C\tilde{C}$ is exact.

Next we show that (3) implies (2). From (3.9) we know that Φ_M is bijective for any $M \in \mathbf{M}_A^C(\psi)$. Therefore for any exact sequence $M_1 \to M_2 \to M_3$ of objects in $\mathbf{M}_A^C(\psi)$ there is an exact sequence $(M_1\widehat{\Box}_C\tilde{C})\hat{\otimes}_{\tilde{A}}A \to (M_2\widehat{\Box}_C\tilde{C})\hat{\otimes}_{\tilde{A}}A \to (M_3\widehat{\Box}_C\tilde{C})\hat{\otimes}_{\tilde{A}}A$. Since $-\hat{\otimes}_{\tilde{A}}A$ reflects exact sequences, there is an exact sequence $M_1\widehat{\Box}_C\tilde{C} \to M_2\widehat{\Box}_C\tilde{C} \to M_3\widehat{\Box}_C\tilde{C}$, i.e., $-\widehat{\Box}_C\tilde{C}$ is exact as required. Similarly one shows that (4) implies (2). $\Box$

Theorem 3.10 applied to entwining structures of Example 3.1(3) and measurings coming from morphisms of entwining structures gives [7, Theorem 2.8] (and Proposition 3.6 gives [7, Theorem 1.3]). Furthermore we obtain the following generalisation of [26, Theorem 3.7]

COROLLARY 3.11 *For an entwining structure $(A, C)_\psi$ the following are equivalent:*

(1) $A(B)^C$ is a C-Galois extension with the canonical entwining map ψ and A is faithfully flat as a left B-module.

(2) $A \in \mathbf{M}_A^C(\psi)$ and the functor $\mathbf{M}_A^C(\psi) \to \mathbf{M}_B$, $M \mapsto M_0$ is an equivalence.

Proof. A left B-module A is faithfully flat iff the functor $- \otimes_B A$ is faithfully exact, hence the assertion follows by applying Theorem 3.10 to Example 3.9. □

Corollary 3.12 *If $(A,C)_\psi$ is the canonical entwining structure associated to a cleft C-Galois extension $A(B)^C$ then $\mathbf{M}_A^C(\psi)$ is equivalent to $\mathbf{M}_B$.*

Proof. $A \cong B \otimes C$ as objects in ${}_B\mathbf{Mod}^C$, so A is a faithfully flat left B-module. □

The following proposition, which is an $(A,C)_\psi$-module version of [12, Theorem 2.11], gives a criterion for faithful flatness of a C-Galois extension $A(B)^C$

Proposition 3.13 *Let $A(B)^C$ be a C-Galois extension and assume that there exists a linear map $\varphi : C \to A$ such that $1_{(0)}\varphi(1_{(1)}) = 1$ and*

$$\psi(c_{(1)} \otimes \varphi(c_{(2)})) = \varphi(c)1_{(0)} \otimes 1_{(1)}, \qquad \forall c \in C.$$

If either A is flat as a left B-module or for all $b \in B$ and $c \in C$, $b_\alpha\varphi(c^\alpha) = \varphi(c)b$ then A is faithfully flat as a left B-module.

Proof. We are in the setting of Example 3.9, thus it suffices to show that the functor $M \mapsto M_0$ is an equivalence and then use Corollary 3.11 to deduce the assertion. First notice that for all $a \in A$, we have $a_{(0)}\varphi(a_{(1)}) \in B$. For any right B module V the adjunction $\Psi_V : V \to (V \otimes_B A)_0$ is simply $v \mapsto v \otimes_B 1$ and has the inverse $\Psi_V^{-1} : (V \otimes_B A)_0 \to V$, $\sum_i v^i \otimes_B a^i \mapsto \sum_i v^i \cdot (a^i{}_{(0)}\varphi(a^i{}_{(1)}))$. Now consider the commutative diagram (3.9) for the measuring of Example 3.9. If A is a flat left B-module then the top sequence is exact and thus Φ_M is bijective. Therefore the equivalence of categories holds in this case. On the other hand we have an exact sequence

$$0 \to \overline{M \otimes_B A} \to M \otimes_B A \to (M \otimes C) \otimes_B A.$$

The elements of $\overline{M \otimes_B A}$ are characterised by the property $\sum_i \Delta_M(m^i) \otimes_B a^i = \sum_i m^i \cdot 1_{(0)} \otimes 1_{(1)} \otimes_B a^i$. $M_0 \otimes_B A$ is included in $\overline{M \otimes_B A}$ canonically. Consider the map $M \to M_0$, $m \mapsto m_{(0)} \cdot \varphi(m_{(1)})$. If for all $b \in B$ and $c \in C$ $b_\alpha \varphi(c^\alpha) = \varphi(c)b$ then this map is a right B-module homomorphism. This implies that the map $\overline{M \otimes_B A} \to M_0 \otimes_B A$, $\sum_i m^i \otimes_B a^i \mapsto \sum_i m^i{}_{(0)} \cdot \varphi(m^i{}_{(1)}) \otimes_B a^i$ is well-defined. It is an easy exercise to verify that this map is an inverse of the canonical inclusion $M_0 \otimes_B A \hookrightarrow \overline{M \otimes_B A}$. Thus we conclude that also in this case the top sequence in (3.9) (for a measuring of Example 3.9) is exact so that the functor $M \mapsto M_0$ is an equivalence. $\square$

4. Associated modules $A\Box_C V$

In this section we construct the left B-module E for each C-Galois extension A of B and a left C-comodule V, and then study the properties of E. This construction is a very general algebraic dualisation of associating of a fibre bundle to a principal bundle.

First recall the definition of a cotensor product [22]. Let C be a coalgebra and $V_R \in \mathbf{M}^C$, $V_L \in {}^C\mathbf{M}$. The cotensor product $V_R \Box_C V_L$ is defined by the exact sequence

$$0 \longrightarrow V_R \Box_C V_L \hookrightarrow V_R \otimes V_L \stackrel{\ell_{V_R V_L}}{\longrightarrow} V_R \otimes C \otimes V_L,$$

where $\ell_{V_R V_L}$ is the coaction equalising map $\ell_{V_R V_L} = \Delta_{V_R} \otimes V_L - V_R \otimes {}_{V_L}\Delta$.

Definition 4.1 *Let $A(B)^C$ be a C-Galois extension. A left B-module E is called a* left module associated to $A(B)^C$ *iff there exists a left C-comodule V such that $E = A\Box_C V$. In this case E is denoted by $E(A(B)^C; V)$.*

Since $A(B)^C$ can be viewed as an object dual to a (generalised) principal bundle and V is dual to a representation of a "structure group", E can be viewed as an object dual to a fibre bundle associated to a principal bundle. In particular, [26, Lemma 3.1(i)] implies that a *quantum fibre bundle* of [3, Definition A.3] associated to a Hopf-Galois extension $A(B)^H$ is a left module associated to $A(B)^H$ provided the antipode in H is bijective. As should be expected, $E(A(B)^C; C) = A(B)^C$, since $A \cong A\Box_C C$ via $\Delta_A : A \to A\Box_C C$

and $A \otimes \varepsilon : A \Box_C C \to A$ (cf. [14, Lemma 2.2*]). Furthermore, if $A(B)^C$ is a cleft C-Galois extension then $A \cong B \otimes C$ in ${}_B\mathbf{Mod}^C$, and for any left C-comodule V we have $E = (B \otimes C)\Box_C V \cong B \otimes V$ in ${}_B\mathbf{M}$. This last statement reflects the classical fact that every fibre bundle associated to a trivial principal bundle is trivial.

Definition 4.2 *Let E be a left module associated to $A(B)^C$. Any left B-module map $s : E \to B$ is called a* cross-section *of E.*

The space $\mathrm{Hom}_{B-}(E, B)$ of all cross-sections of $E(A(B)^C; V)$ has a natural right B-module structure given by $(s \cdot b)(x) = s(x)b$, for all $b \in B$, $x \in E$. Let for a given C-Galois extension $A(B)^C$ and a left C-comodule V, $\mathrm{Hom}_\psi(V, A)$ denote the space of all linear maps $V \to A$ such that for all $v \in V$

$$\psi(v_{(-1)} \otimes \varphi(v_{(0)})) = \varphi(v)\Delta_A(1), \tag{4.10}$$

where $\psi : C \otimes A \to A \otimes C$ is the canonical entwining map associated to $A(B)^C$. For all $\varphi \in \mathrm{Hom}_\psi(V, A)$, $b \in B$ and $v \in V$ we have

$$\begin{aligned}
\psi(v_{(-1)} \otimes \varphi(v_{(0)})b) &= \varphi(v_{(0)})_\alpha \psi(v_{(-1)}{}^\alpha \otimes b) && \text{(by (2.1))} \\
&= \varphi(v)1_{(0)}\psi(1_{(1)} \otimes b) && (\varphi \in \mathrm{Hom}_\psi(V, A)) \\
&= (\varphi(v)b)1_{(0)} \otimes 1_{(1)}. && (b \in B)
\end{aligned}$$

This implies that $\mathrm{Hom}_\psi(V, A)$ is a right B-module with the action given by $(\varphi \cdot b)(v) = \varphi(v)b$. The following theorem reflects the classical equivalence between cross-sections of a fibre bundle and equivariant functions on it (cf. [16, 4.8.1]).

Theorem 4.3 *Let $E = E(A(B)^C; V)$. If either A is a flat right B-module or else V is a coflat left C-comodule, then the right B-modules $\mathrm{Hom}_{B-}(E, B)$ and $\mathrm{Hom}_\psi(V, A)$ are isomorphic to each other.*

Proof. The flatness (coflatness) assumption implies that $(A \otimes_B A)\Box_C V \cong A \otimes_B (A \Box_C V)$, canonically (cf. [26, p. 172]). Thus there is a left A-module isomorphism $\rho : A \otimes_B$

$E \to A\otimes V$, obtained as a composition of $can\Box_C V$ with the canonical isomorphism $A\otimes C\Box_C V \xrightarrow{\sim} A\otimes V$, i.e., $\rho = \mu\otimes V$, $\rho^{-1} = (can^{-1}\otimes V)\circ(A\otimes {}_V\Delta)$. Following [12], apply $\mathrm{Hom}_{A-}(-,A)$ to ρ to deduce the isomorphism

$$\theta : \mathrm{Hom}(V,A) \xrightarrow{\sim} \mathrm{Hom}_{B-}(E,A), \qquad \theta(\varphi)(\sum_i a^i\otimes v^i) = \sum_i a^i\varphi(v^i).$$

Notice that θ is a right B-module map. For any $\varphi\in\mathrm{Hom}(V,A)$, $x=\sum_i a^i\otimes v^i\in E$

$$\Delta_A(\theta(\varphi)(x)) = \sum_i \Delta_A(a^i\varphi(v^i)) = \sum_i {a^i}_{(0)}\psi({a^i}_{(1)}\otimes\varphi(v^i)) = \sum_i a^i\psi({v^i}_{(-1)}\otimes\varphi({v^i}_{(0)})).$$

Therefore $\theta(\varphi)\in\mathrm{Hom}_{B-}(E,B)$ iff

$$\sum_i a^i\psi({v^i}_{(-1)}\otimes\varphi({v^i}_{(0)})) = \sum_i a^i\varphi(v^i)1_{(0)}\otimes 1_{(1)}, \tag{4.11}$$

since $B=A_0$ by Example 3.9. Clearly, (4.10) implies (4.11). Applying (4.11) to $\rho^{-1}(1\otimes v)$ one easily finds that (4.11) implies (4.10). Therefore θ restricts to $\mathrm{Hom}_\psi(V,A)\xrightarrow{\sim}\mathrm{Hom}_{B-}(E,B)$ as a right B-module map. $\Box$

Viewing a C-Galois extension as a left module associated to itself, one can state Proposition 3.13 as follows

Proposition 4.4 *If a C-Galois extension $A(B)^C$ admits a unital B-bimodule map $s : A\to B$ then A is faithfully flat as a left B-module.*

Proof. We view $A(B)^C$ as $E(A(B)^C;C)$. Then $\varphi\in\mathrm{Hom}_\psi(C,A)$ iff for all $c\in C$, $\psi(c_{(1)}\otimes\varphi(c_{(2)})) = \varphi(c)\Delta_A(1)$. The cross-sections are simply left B-module maps $A\to B$. Since C is coflat in ${}^C\mathbf{M}$, by Theorem 4.3, there is an isomorphism of right B-modules $\theta : \mathrm{Hom}_\psi(C,A)\xrightarrow{\sim}\mathrm{Hom}_{B-}(A,B)$, $\theta(\varphi)(a) = a_{(0)}\varphi(a_{(1)})$ and $\theta^{-1}(s)(c) = c^{(1)}s(c^{(2)})$, where $c^{(1)}\otimes_B c^{(2)} = can^{-1}(1\otimes c)$. We now assume that there is a unital B-bimodule map $s:A\to B$, and let $\varphi=\theta^{-1}(s)$. Since s is unital $1_{(0)}\varphi(1_{(1)}) := 1_{(0)}\theta^{-1}(s)(1_{(1)}) = 1_{(0)}{1_{(1)}}^{(1)}s({1_{(1)}}^{(2)}) = s(1) = 1$. Furthermore for any $b\in B$ and $c\in C$

$$\begin{aligned} b_\alpha\varphi(c^\alpha) &= c^{(1)}(c^{(2)}b)_{(0)}\varphi((c^{(2)}b)_{(1)}) = c^{(1)}(c^{(2)}b)_{(0)}{(c^{(2)}b)_{(1)}}^{(1)}s({(c^{(2)}b)_{(1)}}^{(2)})\\ &= c^{(1)}s(c^{(2)}b) = \varphi(c)b, \end{aligned}$$

where we used the definition of the canonical entwining structure to derive the first equality, then the following property of the translation map (cf. [27, Remark 3.4])

$$a_{(0)}a_{(1)}{}^{(1)} \otimes_B a_{(1)}{}^{(2)} = 1 \otimes_B a, \qquad \forall a \in A, \tag{4.12}$$

to derive the third one, and the right B-module property of s to obtain the last equality. Thus we conclude that φ satisfies all the assumptions of Proposition 3.13 and hence the assertion follows. $\square$

Next we establish the equivalence between a certain class of cross-sections of $A(B)^C$ and cleaving maps (cf. [21] for an interesting special case). This reflects the classical equivalence between cross-sections and trivialisations of a principal bundle (cf. [16, 4.8.3])

Proposition 4.5 *A C-Galois extension $A(B)^C$ is cleft if and only if there exists a cross-section $s \in \mathrm{Hom}_{B-}(A, B)$ such that $\hat{s} := (s \otimes C) \circ \Delta_A : A \to B \otimes C$ is a bijection.*

Proof. Assume first that $A(B)^C$ is cleft with a cleaving map $\Phi : C \to A$. Then (2.4) implies that $\Phi^{-1} \in \mathrm{Hom}_\psi(C, A)$. By Theorem 4.3 there is a cross-section $s = \theta(\Phi^{-1})$. Explicitly $s(a) = a_{(0)}\Phi^{-1}(a_{(1)})$. The induced map $\hat{s}$ is thus $\hat{s}(a) = a_{(0)}\Phi^{-1}(a_{(1)}) \otimes a_{(2)}$ and has the inverse $b \otimes c \mapsto b\Phi(c)$ as in Proposition 2.3.

Assume now that there exists $s \in \mathrm{Hom}_{B-}(A, B)$ such that the map $\hat{s}$ is bijective. Since s is a left B-module map, $\hat{s}$ is a morphism in ${}_B\mathbf{Mod}^C$, where $B \otimes C$ is viewed as an object in ${}_B\mathbf{Mod}^C$ via $\mu \otimes C$ and $B \otimes \Delta$. This implies that also $\hat{s}^{-1}$ is a morphism in ${}_B\mathbf{Mod}^C$. Note also that $s = (B \otimes \varepsilon) \circ \hat{s}$. Using Theorem 4.3 we consider $\tilde{\Phi} \in \mathrm{Hom}_\psi(C, A)$ given by $\tilde{\Phi} = \theta^{-1}(s)$, and also a map $\Phi : C \to A$, $\Phi : c \mapsto \hat{s}^{-1}(1 \otimes c)$. We will show that Φ and $\tilde{\Phi}$ are convolution inverses to each other. For any $c \in C$ one has

$$\begin{aligned}
\Phi(c_{(1)})\tilde{\Phi}(c_{(2)}) &= \hat{s}^{-1}(1 \otimes c_{(1)})\theta^{-1}(s)(c_{(2)}) = \hat{s}^{-1}(1 \otimes c_{(1)})c_{(2)}{}^{(1)}s(c_{(2)}{}^{(2)}) \\
&= \hat{s}^{-1}(1 \otimes c)_{(0)}\hat{s}^{-1}(1 \otimes c)_{(1)}{}^{(1)}s(\hat{s}^{-1}(1 \otimes c)_{(1)}{}^{(2)}) \qquad (\hat{s} \text{ is } C\text{-colinear}) \\
&= s(\hat{s}^{-1}(1 \otimes c)) \qquad (\text{by } (4.12)) \\
&= ((B \otimes \varepsilon) \circ \hat{s} \circ \hat{s}^{-1})(1 \otimes c) = \varepsilon(c),
\end{aligned}$$

where $c^{(1)} \otimes_B c^{(2)} = can^{-1}(1 \otimes c)$ as before. On the other hand, for all $a \in A$, one finds

$$a_{(0)}\tilde{\Phi}(a_{(1)})\Phi(a_{(2)}) = a_{(0)}\theta^{-1}(s)(a_{(1)})\hat{s}^{-1}(1 \otimes a_{(2)}) = a_{(0)}{a_{(1)}}^{(1)}s({a_{(1)}}^{(2)})\hat{s}^{-1}(1 \otimes a_{(2)}).$$

Making use of (4.12) and the fact that $\hat{s}^{-1}$ is a left B-module map one concludes

$$a_{(0)}\tilde{\Phi}(a_{(1)})\Phi(a_{(2)}) = s(a_{(0)})\hat{s}^{-1}(1 \otimes a_{(1)}) = \hat{s}^{-1}(s(a_{(0)}) \otimes a_{(1)}) = \hat{s}^{-1}(\hat{s}(a)) = a.$$

Finally, using the definition of the translation map one obtains for all $c \in C$

$$\tilde{\Phi}(c_{(1)})\Phi(c_{(2)}) = c^{(1)}{c^{(2)}}_{(0)}\tilde{\Phi}({c^{(2)}}_{(1)})\Phi({c^{(2)}}_{(2)}) = c^{(1)}c^{(2)} = \varepsilon(c).$$

From the fact that $\tilde{\Phi} \in \mathrm{Hom}_\psi(C, A)$ it is clear that its convolution inverse Φ is a right C-comodule map. Thus we have proven that $A(B)^C$ is cleft. $\square$

Since $\hat{s}$ is a morphism in ${}_B\mathbf{Mod}^C$, Proposition 4.5 states that a C-Galois extension $A(B)^C$ is cleft if and only if $A \cong B \otimes C$ as objects in ${}_B\mathbf{Mod}^C$ (cf. [11, Theorem 9])

5. Associated modules $(V \otimes A)_0$

In this section we construct the right B-module $\bar{E}$ for each C-Galois extension A of B and a right C-comodule V, and then study properties of $\bar{E}$. This construction is another algebraic dualisation of associating of a fibre bundle to a principal bundle.

Definition 5.1 *Let $A(B)^C$ be a coalgebra Galois extension. A right B-module $\bar{E}$ is called a* right module associated to $A(B)^C$ *iff there exists a right C-comodule V such that $\bar{E} = (V \otimes A)_0$ (cf. Example 3.9), where $V \otimes A$ is an $(A, C)_\psi$-module of the canonical entwining structure as in Corollary 3.4(2). In this case $\bar{E}$ is denoted by $\bar{E}(V; A(B)^C)$.*

The right B-module $\bar{E}$ consists of all elements $\sum_i v^i \otimes a^i \in V \otimes A$ such that $\sum_i {v^i}_{(0)} \otimes \psi({v^i}_{(1)} \otimes a^i) = \sum_i v^i \otimes a^i 1_{(0)} \otimes 1_{(1)}$. In the case of a Hopf-Galois extension, if V is a right comodule algebra the above definition of $\bar{E}$ coincides with the definition of a quantum associated fibre bundle in [15, Definition A.1]. Thus, similarly as in Section 5, the constructed right module $\bar{E}$ can be viewed geometrically as an object dual to a fibre bundle associated to a principal bundle of which $A(B)^C$ is a dual object.

Proposition 5.2 *If $A(B)^C$ is cleft then for all right comodules V, $\bar{E} = (V \otimes A)_0$ is isomorphic to $V \otimes B$ as a right B-module.*

Proof. The isomorphism and its inverse are:

$$V \otimes B \to \bar{E}, \qquad v \otimes b \mapsto v_{(0)} \otimes \Phi^{-1}(v_{(1)})b,$$

$$\bar{E} \to V \otimes B, \qquad \sum_i v^i \otimes a^i \mapsto v^i{}_{(0)} \otimes \Phi(v^i{}_{(1)})a^i,$$

where Φ is a cleaving map. To see that the output of the first of these maps is in $\bar{E}$ we compute

$$\begin{aligned} v_{(0)} \otimes \psi(v_{(1)} \otimes \Phi^{-1}(v_{(2)})b) &= v_{(0)} \otimes \Phi^{-1}(v_{(1)})1_{(0)}\psi(1_{(1)} \otimes b) && \text{(by (2.1) and (2.4))} \\ &= v_{(0)} \otimes \Phi^{-1}(v_{(1)})b1_{(0)} \otimes 1_{(1)}. \end{aligned}$$

To verify that the output of the second of the above maps is in $V \otimes B$ we use the fact that A is an $(A, C)_\psi$-module and that Φ is a right C-comodule map to compute

$$\sum_i v^i{}_{(0)}\Delta_A(\Phi(v^i{}_{(1)})a^i) = \sum_i v^i{}_{(0)}\Phi(v^i{}_{(1)})\psi(v^i{}_{(2)} \otimes a^i) = \sum_i v^i{}_{(0)}\Phi(v^i{}_{(1)})a^i 1_{(0)} \otimes 1_{(1)}.$$

It is obvious that the above maps are inverses to each other and that they are right B-module homomorphisms. $\square$

Definition 5.3 *Let $\bar{E}$ be a right module associated to $A(B)^C$. Any right B-module map $s : \bar{E} \to B$ is called a* cross-section *of $\bar{E}$.*

The space $\mathrm{Hom}_{-B}(\bar{E}, B)$ of cross-sections of $\bar{E}$ has a natural left B-module structure. Since Δ_A is left linear over B, the space $\mathrm{Hom}^{-C}(V, A)$ of all right C-comodule maps $\varphi : V \to A$ has a left B-module structure $b \cdot \varphi : v \mapsto b\varphi(v)$.

Theorem 5.4 *For any $\bar{E}(V; A(B)^C)$, if A is faithfully flat as a left B-module then $\mathrm{Hom}^{-C}(V, A) \cong \mathrm{Hom}_{-B}(\bar{E}, B)$ as left B-modules.*

Proof. By Corollary 3.7(2) there is a natural isomorphism $\zeta_{A,V} : \mathrm{Hom}_{-A}^{-C}(V \otimes A, A) \xrightarrow{\sim} \mathrm{Hom}^{-C}(V, A)$, given by $\zeta_{A,V}(s)(v) = s(v\otimes 1)$, $\zeta_{A,V}^{-1}(\varphi)(v\otimes a) = \varphi(v)a$. It is an easy exercise to verify that $\zeta_{A,V}$ preserves the left B-module structure. If A is faithfully flat as a left B-module then by Corollary 3.11 the functor $\mathbf{M}_A^C(\psi) \to \mathbf{M}_B$, $M \mapsto M_0$ is an equivalence. Therefore the space $\mathrm{Hom}_{-A}^{-C}(V\otimes A, A)$ is isomorphic to the space of morphisms $(V\otimes A)_0 \to B$ in $\mathbf{M}_B$. Since the latter is precisely $\mathrm{Hom}_{-B}(\bar{E}, B)$ and the equivalence preserves the left B-module structure introduced on the spaces of morphisms, we conclude that there is an isomorphism of left B-modules $\mathrm{Hom}_{-B}(\bar{E}, B) \cong \mathrm{Hom}^{-C}(V, A)$. □

6. Bijectivity of ψ and the relationship between E and $\bar{E}$

In this section we study a relationship between left and right modules associated to a C-Galois extension. In particular we show that if V is finite dimensional then $E(A(B)^C; V)$ can be identified with the module of cross-sections of $\bar{E}(V^*; A(B)^C)$, and vice versa. Also we show that if the canonical entwining map ψ is bijective then $A\Box_C V$ can be identified with the space of coinvariants ${}_0(A \otimes V)$ of the left coaction of C on $A \otimes V$. The module ${}_0(A \otimes V)$ can be then viewed as $\bar{E}$ for the left C-Galois extension A of B.

If V is a finite-dimensional left C-comodule, the dual vector space V^* is viewed as a right C-comodule via $\langle {v^*}_{(0)}, v\rangle {v^*}_{(1)} = v_{(-1)}\langle v^*, v_{(0)}\rangle$, for all $v \in V$, $v^* \in V^*$, where $\langle \cdot, \cdot\rangle : V^* \otimes V \to k$ denotes the non-degenerate pairing.

Proposition 6.1 *Let $A(B)^C$ be a C-Galois extension, V be a finite dimensional left C-comodule, and V^* the dual right C-comodule. Then:*

(1) The left B-modules $E = A\Box_C V$ and $\mathrm{Hom}^{-C}(V^*, A)$ *are isomorphic to each other.*

(2) The right B-modules $\bar{E} = (V^* \otimes A)_0$ and $\mathrm{Hom}_\psi(V, A)$ *are isomorphic to each other.*

Proof. (1) It is well-known that the vector spaces $\mathrm{Hom}^{-C}(V^*, A)$ and $A\Box_C V$ are isomorphic to each other with the isomorphism $\varphi \mapsto \sum_i a^i \otimes v^i$, such that for all $v^* \in V^*$, $\varphi(v^*) = \sum_i a^i\langle v^*, v^i\rangle$. Clearly, this is also an isomorphism of left B-modules.

(2) We identify $\varphi \in \mathrm{Hom}(V, A)$ with $\sum_i v^{*i} \otimes a^i \in V^* \otimes A$ via $\varphi(v) = \sum_i \langle v^{*i}, v\rangle a^i$. Clearly this identification is an isomorphism of right B-modules. We have

$$\psi(v_{(-1)} \otimes \varphi(v_{(0)})) = \sum_i \langle v^{*i}, v_{(0)}\rangle \psi(v_{(-1)} \otimes a^i) = \sum_i \langle {v^{*i}}_{(0)}, v\rangle \psi({v^{*i}}_{(1)} \otimes a^i).$$

On the other hand $\varphi(v)1_{(0)} \otimes 1_{(1)} = \sum_i \langle v^{*i}, v\rangle a^i 1_{(0)} \otimes 1_{(1)}$ which implies that $\varphi \in \mathrm{Hom}_\psi(V, A)$ if and only if $\sum_i v^{*i} \otimes a^i \in (V^* \otimes A)_0 = \bar{E}$. $\square$

Corollary 6.2 *(1) Let $E = E(A(B)^C; V)$. If V is finite dimensional and A is faithfully flat as a left B-module then the left B-modules E and $\mathrm{Hom}_{-B}(\mathrm{Hom}_\psi(V, A), B)$ are isomorphic to each other.*

(2) Let $\bar{E} = \bar{E}(V; A(B)^C)$ with finite dimensional V. If either A is flat as a right B-module or else V is coflat as a left C-comodule, then the right B-modules $\bar{E}$ and $\mathrm{Hom}_{B-}(\mathrm{Hom}^{-C}(V, A), B)$ are isomorphic to each other.

Proof. (1) By Proposition 6.1 one identifies E with $\mathrm{Hom}^{-C}(V^*, A)$ and also $\mathrm{Hom}_\psi(V, A)$ with $\bar{E}(V^*; A(B)^C)$, and then applies Theorem 5.4 to deduce the isomorphism of left B-modules.

(2) By Proposition 6.1 one identifies $\bar{E}$ with $\mathrm{Hom}_\psi(V^*, A)$ which by Theorem 4.3 is isomorphic to the right module of cross-sections $\mathrm{Hom}_{B-}(A\Box_C V^*, B)$. Then one applies Proposition 6.1 again to deduce the required isomorphism. $\square$

Remark 6.3 For any left module $E(A(B)^C; V)$ there is a left B-module map $E \to \mathrm{Hom}_{-B}(\mathrm{Hom}_\psi(V, A), B)$ given by $\sum_i a^i \otimes v^i \mapsto s$ where $s(\varphi) = \sum_i a^i \varphi(v^i)$. Similarly for any $\bar{E}(V; A(B)^C)$ there is a right B-module map $\bar{E} \to \mathrm{Hom}_{B-}(\mathrm{Hom}^{-C}(V, A), B)$ given by $\sum_i v^i \otimes a^i \mapsto s$, where $s(\varphi) = \sum_i \varphi(v^i) a^i$. $\diamond$

The remaining part of this section is devoted to studies of the relationship between E and $\bar{E}$ in the case when the canonical entwining map is bijective.

Example 6.4 (1) Let H, A, C and ψ be as in Example 3.1(3). If the antipode S in H is bijective then ψ is bijective. Explicitly $\psi^{-1}(a\otimes c) = c\cdot S^{-1}a_{(1)}\otimes a_{(0)}$, $\forall a\in A, c\in C$.

(2) For a Hopf-Galois extension $A(B)^H$, the canonical entwining map $\psi : h\otimes a\mapsto a_{(0)}\otimes ha_{(1)}$ is bijective if and only if the antipode in H is bijective.

Proof. (1) is proven by a straightforward computation. To prove (2) consider the linear map $\psi_H : H\otimes H\to H\otimes H$, $h\otimes h'\mapsto h'_{(1)}\otimes hh'_{(2)}$. It is well-known that ψ_H is bijective if and only if the antipode is bijective. Notice that $A\otimes_B\psi = (can^{-1}\otimes H)\circ(A\otimes\psi_H)\circ(can\otimes H)$. This completes the proof. □

Lemma 6.5 *Let $A(B)^C$ be a C-Galois extension and assume that the canonical entwining map ψ is bijective. Then:*

(1) A is a left C-comodule with the coaction ${}_A\Delta(a) = \psi^{-1}(a1_{(0)}\otimes 1_{(1)})$.

(2) The canonical right A-module left C-comodule map $can_L : A\otimes_B A\to C\otimes A$, $a\otimes a'\mapsto {}_A\Delta(a)a'$ is bijective.

(3) The algebra B is isomorphic to

$$\bar{B} := \{b\in A \mid \forall a\in A,\ {}_A\Delta(ab) =_A \Delta(a)b\} = \{b\in A \mid {}_A\Delta(b) = \psi^{-1}(1_{(0)}\otimes 1_{(1)})b\}.$$

Proof. (1) is proven in the way analogous to the proof that ψ induces a right C-coaction on A via $\Delta_A(a) = 1_{(0)}\psi(1_{(1)}\otimes a)$. To prove (2) notice that $can_L = \psi^{-1}\circ can$ thus it is well defined and a bijection. Next take $a\in A$ and $b\in B$, then ${}_A\Delta(ab) = \mathrm{can}_L(ab\otimes_B 1) = can_L(a\otimes_B b) = \mathrm{can}_L(a\otimes_B 1)b = {}_A\Delta(a)b$, i.e. $b\in\bar{B}$. On the other hand take $\bar{b}\in\bar{B}$. Then ${}_A\Delta(\bar{b}) = \psi^{-1}(1_{(0)}\otimes 1_{(1)})\bar{b}$, i.e. $\psi^{-1}(\bar{b}1_{(0)}\otimes 1_{(1)}) = \psi^{-1}(1_{(0)}\otimes 1_{(1)})\bar{b}$. Applying ψ one obtains $\bar{b}1_{(0)}\otimes 1_{(1)} = 1_{(0)}\psi(1_{(1)}\otimes\bar{b}) = \Delta_A(\bar{b})$, i.e. $\bar{b}\in B$ by Example 3.9. □

Having $\psi^{-1} : A\otimes C\to C\otimes A$ one can consider category ${}_A^C\mathbf{M}(\psi^{-1})$ the objects of which are left A-modules and left C-comodules M such that

$${}_M\Delta(a\cdot m) = \psi^{-1}(a\otimes m_{(-1)})\cdot m_{(0)}.$$

The morphisms are left A-module left C-comodule maps. In particular, if the canonical entwining map ψ of $A(B)^C$ is bijective, then $A \in {}^C_A\mathbf{M}(\psi^{-1})$. One also defines a functor ${}^C_A\mathbf{M}(\psi^{-1}) \to {}_B\mathbf{M}$, $M \mapsto {}_0M$, where ${}_0M := \{m \in M \mid {}_M\Delta(m) = \psi^{-1}(1_{(0)} \otimes 1_{(1)}) \cdot m\}$. If V is a left C-comodule then $A \otimes V$ is an object in ${}^C_A\mathbf{M}(\psi^{-1})$, where the left coaction is given by ${}_{A\otimes V}\Delta(a \otimes v) = \psi^{-1}(a \otimes v_{(-1)}) \otimes v_{(0)}$ and the action is $\mu \otimes V$. Indeed,

$$\begin{aligned} {}_{A\otimes V}\Delta(a'a \otimes v) &= \psi^{-1}(a'a \otimes v_{(-1)}) \otimes v_{(0)} = v_{(-1)}{}_{\alpha\beta} \otimes a'^{\beta} a^{\alpha} \otimes v_{(0)} \\ &= (a \otimes v)_{(-1)}{}_{\beta} \otimes a'^{\beta} \cdot (a \otimes v)_{(0)} \\ &= \psi^{-1}(a' \otimes (a \otimes v)_{(-1)}) \cdot (a \otimes v)_{(0)}, \end{aligned}$$

where $\psi^{-1}(a \otimes c) = c_\alpha \otimes a^\alpha$. This implies that ${}_0(A \otimes V)$ is a left B-module.

Proposition 6.6 *If the canonical entwining map ψ of $A(B)^C$ is bijective then:*

(1) For any left C-comodule V, the left B-modules ${}_0(A\otimes V)$ and $A\Box_C V$ are isomorphic to each other.

(2) For any right C-comodule V, the right B-modules $(V \otimes A)_0$ and $V\Box_C A$ are isomorphic to each other.

Proof. (1) Take $\sum_i a^i \otimes v^i \in {}_0(A \otimes V)$. It means that $\sum_i \psi^{-1}(a^i \otimes v^i{}_{(-1)}) \otimes v^i{}_{(0)} = \sum_i \psi^{-1}(1_{(0)} \otimes 1_{(1)}) a^i \otimes v^i$. Applying ψ one obtains

$$\sum_i a^i \otimes v^i{}_{(-1)} \otimes v^i{}_{(0)} = \sum_i 1_{(0)}\psi(1_{(1)} \otimes a^i) \otimes v^i = \sum_i a^i{}_{(0)} \otimes a^i{}_{(1)} \otimes v^i.$$

Therefore $\sum_i a^i \otimes v^i \in A\Box_C V$. To prove the second inclusion one repeats above steps in a reversed order.

The proof of (2) is analogous to (1). □

Lemma 6.5 shows that any right C-Galois extension (i.e. with a right coaction) that has the bijective canonical entwining map, can be viewed equivalently as a left C-Galois extension (i.e. with a left coaction). Then Proposition 6.6 yields that the module E associated to a right C-Galois extension $A(B)^C$ plays the role of $\bar{E}$ when $A(B)^C$ is viewed as a left C-Galois extension. Similarly, the module $\bar{E}$ associated to the right extension corresponds to E when $A(B)^C$ is viewed as a left C-Galois extension.

Appendix. Dual results

In this appendix we give dual version of the results described in Sections 2-6. Dual counterparts of statements given above are numbered with the same numbers decorated with stars. Proofs can be obtained by dualisation and thus are omitted.

Definition 2.1* ([2]) *Let A be an algebra, C a coalgebra and a right A-module with the action μ_C, and $B = C/I_C$, where $I_C \subseteq C$ is given by*

$$I_C = \mathrm{span}\{(c \cdot a)_{(1)} a^*((c \cdot a)_{(2)}) - c_{(1)} a^*(c_{(2)} \cdot a) \;\mid\; c \in C, a \in A, a^* \in A^*\}.$$

We say that C is an algebra Galois coextension *(or A-Galois coextension) of B iff the canonical left C-comodule right A-module map*

$$cocan := (C \otimes \mu_C) \circ (\Delta \otimes A) \;:\; C \otimes A \longrightarrow C \Box_B C$$

is bijective. Here the coaction equalising map ℓ_{CC} is $\ell_{CC} = (C \otimes \pi_C) \circ \Delta - (\pi_C \otimes C) \circ \Delta$, where $\pi_C : C \to B$ is the canonical surjection. Such an A-Galois coextension is denoted by $C(B)_A$.

We refer the reader to [2, Section 3], where it is shown that B is a coalgebra, μ_C is left B-colinear and *cocan* is well-defined. Also in [2] it is shown that every A-Galois coextension $C(B)_A$ induces a unique entwining map $\psi : C \otimes A \to A \otimes C$, $\psi = (\check{\tau} \otimes C) \circ (C \otimes \Delta) \circ cocan$, such that $C \in \mathbf{M}_A^C(\psi)$. Here $\check{\tau} : C \Box_B C \to A$, $\check{\tau} := (\varepsilon \otimes C) \circ cocan^{-1}$ is the *cotranslation map*. This ψ is called the *canonical entwining structure* associated to $C(B)_A$.

A coextension $C(B)_A$ is *cocleft* if there exists a convolution invertible, right A-module map $\Phi : C \to A$ (cf. [21, Definition 2.2]). The fact that $\Phi(c \cdot a) = \Phi(c)a$ implies

$$\mu \circ (A \otimes \Phi^{-1}) \circ \psi = (\Phi^{-1} \otimes \varepsilon \circ \mu_C) \circ (\Delta \otimes A),$$

which, in turn, allows one to prove that $C \cong B \otimes A$ as objects in ${}^B\mathbf{Mod}_A$.

For $C(B)_A$, $\mathrm{Aut}(C(B)_A)$ denotes the group of left B-comodule, right A-module automorphisms of C with the product given by the composition of maps.

THEOREM 2.4* $\mathrm{Aut}(C(B)_A)$ *is isomorphic to the group* $\mathfrak{A}(C)$ *of convolution invertible maps* $f : C \to A$ *such that*

$$\mu \circ (A \otimes f) \circ \psi = \mu \circ (f \otimes A), \tag{2.5*}$$

where ψ *is the canonical entwining map. The product in* $\mathfrak{A}(C)$ *is the convolution product.*

Notice that the condition (2.5*) defining $\mathfrak{A}(C)$ can be also understood as a twisted commutativity condition, since it explicitly reads for all $a \in A$, $c \in C$, $a_\alpha f(c^\alpha) = f(c)a$. If $C(B)_A$ is a cocleft A-Galois coextension, then $\mathfrak{A}(C)$ is isomorphic to the group of convolution invertible maps $\gamma : B \to A$, since $\mathrm{End}^{B-}_{-A}(B \otimes A) \cong \mathrm{Hom}(B, A)$ as algebras.

EXAMPLE 3.9* Assume that C is an object in $\mathbf{M}^C_A(\psi)$, and let B, π_C be as in Definition 2.1*. Then $(A, C)_\psi$ is measured to the trivial entwining structure $(k, B)_\sigma$ by $(\varepsilon \circ \mu_C, \pi_C)$. With this measuring, for all $V \in \mathbf{M}^B_k(\sigma) = \mathbf{M}^B$, $V \widehat{\Box}^C_B = V \Box_B C$, while for all $M \in \mathbf{M}^C_A(\psi)$, $M \hat{\otimes}_A k = M^0 := M/I_M$, where $I_M := \mathrm{span}\{m \cdot a - m_{(0)}\varepsilon(m_{(1)} \cdot a) \mid a \in A,\ m \in M\}$. Notice that $I_M = \mathrm{span}\{(m\cdot a)_{(0)}a^*((m\cdot a)_{(1)}) - m_{(0)}a^*(m_{(1)}\cdot a) \mid a \in A,\ m \in M,\ a^* \in A^*\}$. The measuring $(\varepsilon \circ \mu_C, \pi_C)$ is Galois iff the coextension $C \twoheadrightarrow B$ is Galois.

COROLLARY 3.11* *For an entwining structure* $(A, C)_\psi$ *the following are equivalent:*

(1) $C(B)_A$ *is an* A*-Galois coextension with the canonical entwining map* ψ *and* C *is faithfully coflat as a left* B*-comodule (i.e. the functor* $-\Box_B C$ *is faithfully exact).*

(2) $C \in \mathbf{M}^C_A(\psi)$ *and the functor* $\mathbf{M}^C_A(\psi) \to \mathbf{M}^B$, $M \mapsto M^0$ *is an equivalence.*

COROLLARY 3.12**If* $(A, C)_\psi$ *is the canonical entwining structure associated to a cocleft* A*-Galois coextension* $C(B)_A$ *then* $\mathbf{M}^C_A(\psi)$ *is equivalent to* $\mathbf{M}^B$*.*

PROPOSITION 3.13* *Let* $C(B)_A$ *be an* A*-Galois coextension and assume that there exists a linear map* $\varphi : C \to A$ *such that* $\varepsilon(c_{(1)} \cdot \varphi(c_{(2)})) = \varepsilon(c)$ *and* $a_\alpha \cdot \varphi(c^\alpha) = \varphi(c_{(1)})\varepsilon(c_{(2)} \cdot a)$*. If either* C *is coflat as a left* B*-comodule or for all* $c \in C$, $\varphi(c_{(2)})_\alpha \otimes \pi_C({c_{(1)}}^\alpha) = \varphi(c_{(1)}) \otimes \pi_C(c_{(2)})$*, then* C *is faithfully coflat as a left* B*-comodule.*

Definition 4.1* *Let $C(B)_A$ be an A-Galois coextension. A left B-comodule E is called a* left comodule associated to $C(B)_A$ *iff there exists a left A-module V such that $E = C \otimes_A V$. In this case E is denoted by $E(C(B)_A; V)$.*

For any A-Galois coextension, $E(C(B)_A; A) = C(B)_A$. Also, if $C(B)_A$ is cocleft, then any $E(C(B)_A; V)$ is isomorphic to $B \otimes V$ as a left B-comodule.

Definition 4.2* *Let E be a left comodule associated to $C(B)_A$. Any left B-comodule map $s : B \to E$ is called a* cross-section *of E.*

The space $\mathrm{Hom}^{B-}(B, E)$ of all cross-sections of $E(C(B)_A; V)$ has a natural right B-comodule structure given by $\Delta_{\mathrm{Hom}^{B-}(B,E)}(s) = (s \otimes B) \circ \Delta$. Let for a given A-Galois coextension $C(B)_A$ and a left A-module V, $\mathrm{Hom}^{\psi}(C, V)$ denote the space of all linear maps $\varphi : C \to V$ such that

$$ {}_V\mu \circ (A \otimes \varphi) \circ \psi = (\varphi \otimes \varepsilon \circ \mu_C) \circ (\Delta \otimes A), \tag{4.10*} $$

where $\psi : C \otimes A \to A \otimes C$ is the canonical entwining map associated to $C(B)_A$. The space $\mathrm{Hom}^{\psi}(C, V)$ is a right C-comodule via $(\varphi \otimes \pi_C) \circ \Delta$.

Theorem 4.3* *Let $E = E(C(B)_A; V)$. If V is flat as a left A-module or C is coflat as a right B-comodule, then the right B-comodules* $\mathrm{Hom}^{B-}(B, E)$ *and* $\mathrm{Hom}^{\psi}(C, V)$ *are isomorphic to each other.*

Proposition 4.4* *If an A-Galois coextension $C(B)_A$ admits a counital B-bicomodule map $s : B \to C$ then C is faithfully coflat as a left B-comodule.*

Proposition 4.5* *An A-Galois coextension $C(B)_A$ is cocleft if and only if there exists a cross-section $s \in$* $\mathrm{Hom}^{B-}(B, E)$ *such that $\hat{s} := \mu_A \circ (s \otimes A) : B \otimes A \to C$ is a bijection.*

Proof. Given s with bijective $\hat{s}$, the cocleaving map and its convolution inverse are $\Phi = (\varepsilon \otimes A) \circ \hat{s}^{-1}$, and $\Phi^{-1} = \check{\tau} \circ (C \otimes s \circ \pi_C) \circ \Delta$. $\square$

An A-Galois coextension $C(B)_A$ is cocleft if and only if $C \cong B \otimes A$ in ${}^B\mathbf{Mod}_A$.

Definition 5.1* *Let $C(B)_A$ be an algebra Galois coextension. A right B-comodule $\bar{E}$ is called a* right comodule associated to $C(B)_A$ *iff there exists a right A-module V such that $\bar{E} = (V \otimes C)^0$, where $V \otimes C$ is an $(A, C)_\psi$-module of the canonical entwining structure as in Corollary 3.4(1). In this case $\bar{E}$ is denoted by $\bar{E}(V; C(B)_A)$.*

Proposition 5.2* *If $C(B)_A$ is a cocleft coextension then any $\bar{E}(V; C(B)_A)$ is isomorphic to $V \otimes B$ as a right B-comodule.*

Proof. The isomorphism and its inverse are:

$$V \otimes B \to \bar{E}, \qquad v \otimes b \mapsto \pi_{V\otimes C}(v \cdot \Phi(c_{(1)}) \otimes c_{(2)}), \quad c \in \pi_C^{-1}(b),$$

$$\bar{E} \to V \otimes B, \qquad x \mapsto \sum_i v^i \cdot \Phi^{-1}({c^i}_{(1)}) \otimes \pi_C({c^i}_{(2)}), \quad \sum_i v^i \otimes c^i \in \pi_{V\otimes C}^{-1}(x),$$

where Φ is a cocleaving map. $\square$

Definition 5.3* *Let $\bar{E}$ be a right comodule associated to $C(B)_A$. Any right B-comodule map $s : B \to \bar{E}$ is called a* cross-section *of $\bar{E}$.*

The space $\mathrm{Hom}^{-B}(B, \bar{E})$ of cross-sections of $\bar{E}$ has a natural left B-comodule structure. Let $\mathrm{Hom}_{-A}(C, V)$ denote the space of right A-module maps $\varphi : C \to V$. Since μ_C is left-colinear over B the space $\mathrm{Hom}_{-A}(C, V)$ is a left B-comodule.

Theorem 5.4* *Let $\bar{E} = \bar{E}(V; C(B)_A)$. If C is faithfully coflat as a left B-comodule then $\mathrm{Hom}_{-A}(C, V) \cong \mathrm{Hom}^{-B}(B, \bar{E})$ as left B-comodules.*

Example 6.4* *For a Hopf-Galois coextension $C(B)_H$ the canonical entwining map $\psi : c \otimes h \mapsto h_{(1)} \otimes c \cdot h_{(2)}$ is bijective if and only if the antipode in H is bijective.*

Lemma 6.5* *Let $C(B)_A$ be an A-Galois coextension with the bijective canonical entwining map ψ. Then:*

(1) C is a left A-module with the action ${}_C\mu = (C \otimes \varepsilon \circ \mu_C) \circ (\Delta \otimes A) \circ \psi^{-1}$.

(2) The canonical right C-comodule left A-module map $cocan_L : A \otimes C \to C \Box_B C$, $a \otimes c \mapsto a \cdot c_{(1)} \otimes c_{(2)}$ is bijective.

(3) The coalgebra B is isomorphic to $\bar{B} := C/\bar{I}_C$, where $\bar{I}_C := \mathrm{span}\{a \cdot c - \varepsilon(a \cdot c_{(1)})c_{(2)} \mid \forall a \in A, c \in C\}$.

Proof. Assertion (1) can be proven by direct calculations which, in particular, use the equation $c \cdot a = \varepsilon(c_{(1)} \cdot a_\alpha)c_{(2)}{}^\alpha$, relating μ_C with ψ. To prove (2) one directly verifies that $cocan_L = cocan \circ \psi^{-1}$. To prove (3) one first defines the map $\bar{\iota}_C : A \otimes C \to \bar{I}_C$, $a \otimes c \mapsto a \cdot c - \varepsilon(a \cdot c_{(1)})c_{(2)}$. An easy calculation reveals that $\bar{\iota}_C = -\iota_C \circ \psi^{-1}$, where $\iota_C : C \otimes A \to I_C$, $c \otimes a \mapsto c \cdot a - c_{(0)}\varepsilon(c_{(1)} \cdot a)$, and thus $I_C = \bar{I}_C$, i.e. $B = \bar{B}$. □

If ψ is bijective then $C \in {}_A^C\mathbf{M}(\psi^{-1})$. Therefore one can define a functor ${}_A^C\mathbf{M}(\psi^{-1}) \to {}^B\mathbf{M}$, $M \mapsto {}^0M$, where ${}^0M := M/\bar{I}_M$,

$$\bar{I}_M := \mathrm{span}\{a \cdot m - \varepsilon(a \cdot m_{(-1)})m_{(0)} \mid \forall m \in M, a \in A\}.$$

If V is a left A-module then $C \otimes V$ is an object in ${}_A^C\mathbf{M}(\psi^{-1})$ where the left action is given by ${}_{C\otimes V}\mu = (C \otimes {}_V\mu) \circ (\psi^{-1} \otimes V)$, and the coaction is ${}_{C\otimes V}\Delta(c \otimes v) = \Delta \otimes V$.

Proposition 6.6* *Let $C(B)_A$ be an A-Galois coextension with the bijective canonical entwining map ψ. Then:*

(1) For any left A-module V, the left B-comodules ${}^0(C \otimes V)$ and $C \otimes_A V$ are isomorphic to each other.

(2) For any right A-module V, the right B-comodules $(V \otimes C)^0$ and $V \otimes_A C$ are isomorphic to each other.

Proof. (1) Consider left B-comodule maps $\bar{\iota}_{C\otimes V} : A \otimes C \otimes V \to C \otimes V$, $a \otimes c \otimes v \mapsto a \cdot (c \otimes v) - \varepsilon(a \cdot c_{(1)})c_{(2)} \otimes v$, and $\kappa : C \otimes A \otimes V \to C \otimes V$, $c \otimes a \otimes v \mapsto c \cdot a \otimes v - c \otimes a \cdot v$. An easy calculation shows that $\bar{\iota}_{C\otimes V} = \kappa \circ (\psi^{-1} \otimes V)$. Therefore we have a commutative diagram of B-comodule maps with exact rows:

$$\begin{array}{ccccccc}
A \otimes C \otimes V & \xrightarrow{\bar{\iota}_{C\otimes V}} & C \otimes V & \longrightarrow & {}^0(C \otimes V) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi^{-1}\otimes V} & & \downarrow{\scriptstyle =} & & \downarrow & & \\
C \otimes A \otimes V & \xrightarrow{\kappa} & C \otimes V & \longrightarrow & C \otimes_A V & \longrightarrow & 0.
\end{array}$$

Thus we conclude that ${}^0(C \otimes V) \cong C \otimes_A V$ as left B-comodules.

(2) Follows from (1) by the left-right symmetry. $\square$

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